ficient; C, radiation coefficient; Fo = $a\tau$, Fourier number (dimensionless time) for plate of unit thickness; $(1 - \gamma)$, relative initial temperature; β_i, μ_i, φ_i, D_i, B, k, b, coefficients defined in text; α, heat transfer coefficient; Bi, Biot number; Bi_p, radiative Biot number for plate of unit thickness.

LITERATURE CITED

1. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow (1967).
2. T. Shup, Solution of Engineering Problems on a Computer [in Russian], Moscow (1982).
3. A. V. Kavaderov and Yu. A. Samoilovich, Combustion, Heat Exchange, and Processes of Heating of Metal in Furnaces [in Russian], Sverdlovsk (1963), pp. 14-50.

EXISTENCE OF SOLITARY WAVES IN A PRESTRESSED NONLINEAR THERMOELASTIC MEDIUM WITH DRY FRICTION

M. D. Martynenko and Fam Shi Vin

UDC 539.3

The one-dimensional problem of the solitary wave propagation in a prestressed nonlinear thermoelastic medium with dry friction is analyzed on the basis of a geometrically nonlinear model. An equation is derived for calculating the free energy at which solitary waves can be generated in such a medium. It is shown that the wave velocity depends on the initial state of the medium and on the dry friction law.

1. BASIC EQUATIONS OF THE NONLINEAR THEORY OF THERMOELASTICITY IN THE PRESENCE OF DRY FRICTION FORCES

Let a body obey the laws of the nonlinear theory of thermoelasticity in the presence of dry friction; the analysis of the wave processes in the one-dimensional problem in Lagrangian variables is then reduced to the solution of the following equations [1-4]: a) the equation of motion

$$\frac{\partial}{\partial x}[(1+\varepsilon)\sigma^*] = \rho_0 \frac{\partial^2 u}{\partial t^2} + \operatorname{sgn} v f(|v|)$$

or

$$\frac{\partial^2}{\partial x^2}[(1+\varepsilon)\sigma^*] = \rho_0 \frac{\partial^2 \varepsilon}{\partial t^2} + \frac{\partial}{\partial x}[\operatorname{sgn} v f(|v|)], \tag{1}$$

where $\varepsilon = \partial u/\partial x$, $v = \partial u/\partial t$, f is a continuously differentiable function on the interval $(0; \alpha]$ $(\alpha > 0)$, $f'(v)$, $f(v) > 0$ for $v \in (0; \alpha]$, and $f(0) = 0$; the condition $f(0) = 0$ ensures the differentiability of the function $\operatorname{sign} v\, f(|v|)$ $(v \in [-\alpha; \alpha])$; b) the heat-conduction equation, assuming that

V. I. Lenin Belorussian State University, Minsk.

Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 62, No. 1, pp. 140-145, January, 1992. Original article submitted March 25, 1991.

1062-0125/92/6201-0114$12.50

$$|T - T_0|/T_0 \ll 1 \quad \text{or} \quad |\theta|/T_0 \ll 1. \tag{2}$$

When condition (2) is satisfied, the heat-conduction equation has the form

$$\frac{ds}{\partial t} = k^* \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\frac{1}{(1+\varepsilon)}\right] + \bar{k} f(|v|)|v|. \tag{3}$$

Her $\theta = T - T_0$, $\bar{k} = 1/\rho_0 T_0$, $k^* = k\bar{k}$, and k is a constant.

The relation between the strains and the displacement has the form

$$e = \varepsilon + \frac{1}{2}\varepsilon^2. \tag{4}$$

The governing equations are

$$\sigma^* = \frac{\partial F}{\partial e}, \quad S = -\frac{\partial F}{\partial \theta}, \quad F = F(e, \theta), \tag{5}$$

where F is the free energy.

2. BASIC EQUATIONS OF THE NONLINEAR THEORY OF THERMOELASTICITY FOR A PRESTRESSED MEDIUM WITH DRY FRICTION

Let $\bar{\sigma}^*$, $\bar{e}$, $\bar{\varepsilon}$, $\bar{S}$, and $\bar{\theta}$ be the characteristics of the medium at a given time; let σ^{*0}, e_0, ε_0, S_0, and θ_0 be the same quantities at the initial time; and let σ^*, e, ε, S and θ be their perturbations. We then have

$$\bar{\sigma}^* = \sigma^{*0} + \sigma^*, \quad \bar{e} = e_0 + e, \quad \bar{\varepsilon} = \varepsilon_0 + \varepsilon, \quad \bar{S} = S_0 + S, \quad \bar{\theta} = \theta_0 + \theta. \tag{6}$$

We shall assume that

$$S_0 = \text{const}, \quad \theta_0 = 0, \quad \varepsilon_0 = \text{const}, \quad \sigma^{*0} = \text{const}. \tag{7}$$

From Eqs. (1), (3)-(7) we obtain the equations

$$\frac{\partial^2}{\partial x^2}[(1 + \varepsilon + \varepsilon_0)\sigma^* + \sigma^{*0}\varepsilon] = \rho_0 \frac{\partial^2 \varepsilon}{\partial t^2} + \frac{\partial}{\partial x}[\operatorname{sgn} v f(|v|)], \tag{8}$$

$$\frac{dS}{dt} = k^* \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\frac{1}{(1 + \varepsilon + \varepsilon_0)}\right] + \bar{k} f(|v|)|v|, \tag{9}$$

$$e + e_0 = (\varepsilon + \varepsilon_0) + \frac{1}{2}(\varepsilon + \varepsilon_0)^2, \quad \varepsilon = \partial x/\partial x, \tag{10}$$

$$\sigma^* + \sigma^{*0} = \frac{\partial F}{\partial (e + e_0)}, \quad S + S_0 = -\frac{\partial F}{\partial \theta}, \quad F = F(e + e_0, \theta). \tag{11}$$

3. ONE-SOLITON WAVES IN A PRESTRESSED NONLINEAR THERMOELASTIC MEDIUM WITH DRY FRICTION

We consider the problem of choosing the function F in such a way that the system (8)-(11) will have a solution in the form of solitary strain waves. We must therefore augment Eqs. (8)-(11) with the following equations, which characterize the soliton property [5, 6]:

$$\frac{\partial^2 e}{\partial z^2} - [n^2 + U]e = 0 \tag{12}$$

or

$$\frac{\partial U}{\partial t} - 6U\frac{\partial U}{\partial z} + \frac{\partial^3 U}{\partial z^3} = 0, \tag{13}$$

$$\frac{\partial e}{\partial t} = -4\frac{\partial^3 e}{\partial z^3} + 6U\frac{\partial e}{\partial z} + 3e\frac{\partial U}{\partial z} - qe, \tag{14}$$

where q and n are constants, and z denotes the coordinates of a point of the medium in the initial state:

$$z = (1 + \varepsilon_0)\, x. \tag{15}$$

We seek a solution of Eqs. (8)-(11), (12)-(14) in the form

$$U = U(e), \ \partial e/\partial z = g(e), \tag{16}$$

$$\sigma^* = \sigma^*(e), \ S = S(e), \ \theta = \theta(e). \tag{17}$$

Under the assumption (16) it is readily proved that

$$\partial e/\partial t = h(e), \ h(e) = \pm c g(e), \tag{18}$$

$$e = e(\xi), \ \xi = z \pm ct,$$

where c is a positive constant (the solitary wave velocity). We shall assume that

$$e = e(\xi), \ \xi = z - ct. \tag{19}$$

We then have

$$\frac{\partial f_*}{\partial t} = -C \frac{df_*}{d\xi}, \ \frac{\partial f_*}{\partial z} = \frac{df_*}{d\xi}, \tag{20}$$

where f_* represents any of the functions e, ε, σ^*, θ, S, g, h, v, or u.

It follows from Eqs. (12), (14), and (20) that

$$(2U + c - 4n^2)\frac{de}{d\xi} = e\frac{dU}{d\xi} + qe\ . \tag{21}$$

From Eqs. (13) and (20) we obtain

$$\frac{dU}{d\xi} = (2U^3 + cU^2 + AU + B)^{1/2}, \tag{22}$$

where A and B are constants of integration.

If U = 0, we infer from Eqs. (12), (20), and (21) that

$$q^2 = n^2 (c - 4n^2)^2. \tag{23}$$

Assuming that

$$(2U + c - 4n^2) \leqslant 0 \tag{24}$$

from Eqs. (21) and (32) we have

$$(2U + c - 4n^2)\exp(qI_3) = -4R^2 e^2, \tag{25}$$

where R is a constant of integration, and

$$I_3 = \int \frac{2dU}{(2U + c - 4n^2)(2U^3 + cU^2 + AU + B)^{1/2}} \tag{26}$$

is a Weierstrass elliptic integral of the third kind.

We now consider the case

$$q \neq 0, \ A = B = 0. \tag{27}$$

It follows from Eqs. (23)-(27) that

$$e = \frac{(2n - y)}{2R}\left|\frac{y + \sqrt{c}}{y - \sqrt{c}}\right|^{\frac{n}{\sqrt{c}}}, \tag{28}$$

where

$$y^2 = 2U + c, \ |y| \leqslant 2n. \tag{29}$$

Let

$$c = n^2, \ n < y \leqslant 2n; \tag{30}$$

we then obtain the following expression from Eqs. (28) and (30):

$$e=\frac{(2n-y)}{2R}\left(\frac{y+n}{y-n}\right). \quad (31)$$

On the other hand, from Eqs. (22) and (27) we obtain

$$y=n\,\mathrm{cth}\left[\frac{1}{2}n(p-\xi)\right], \quad (32)$$

where p is a positive constant of integration.

It follows from Eq. (32) that in order to have $n < y \leq 2n$, it is necessary that

$$0\leqslant\xi\leqslant p-\frac{1}{n}\ln 3. \quad (33)$$

The constants of integration R and p are evaluated from the initial conditions $\partial e/\partial t = h_0$, $e = \beta$ at $x = 0$, $t = 0$. We note that $p - (1/n)\ln 3 > 0$ for sufficiently large values of β. From Eq. (31) we obtain

$$\frac{de}{dy}=-\frac{1}{4R}\left(\frac{y+n}{y-n}\right)(y^2-2ny+2n^2)<0\ \forall y. \quad (34)$$

Consequently, e(y) is a monotonically decreasing function, and

$$0\leqslant e\leqslant\beta, \quad (35)$$

$$e=0 \text{ at } y=2n,\ e=\beta \text{ at } y=n\,\mathrm{cth}\left(\frac{1}{2}np\right). \quad (36)$$

The following expression is deduced from Eqs. (31), (32), (34) and the relation $g(e) = (de/dy)(dy/d\xi)$:

$$g(e)=-\frac{n}{2R}\sqrt{(\bar{e}+n)^2+8n^2}\,[\sqrt{(\bar{e}+n)^2+8n^2}+(\bar{e}+3n)]\,[\sqrt{(\bar{e}+n)^2+8n^2}+(\bar{e}+n)]^{-1}, \quad (37)$$

$$\bar{e}=2Re.$$

Let us assume that $\varepsilon \geq 0$ $\varepsilon_0 \geq 0$; it then follows from Eq. (10) that

$$\varepsilon+\varepsilon_0=\sqrt{1+2(e+e_0)}-1. \quad (38)$$

From Eqs. (10), (15), and (20) we obtain

$$v=-(c\varepsilon)/(1+\varepsilon_0). \quad (39)$$

On the basis of Eqs. (38) and (39) we have the relation

$$v=c(1+2e_0)^{-\frac{1}{2}}(1-\sqrt{1+2e})\leqslant 0. \quad (40)$$

Let

$$(1+\varepsilon_0)\sigma^{*0}=\bar{\rho}_0c^2\varepsilon_0-m\int_0^{e_0}\frac{dt}{\bar{g}(t)}-\frac{1}{(1+\varepsilon_0)}\int_0^{e_0}\frac{\bar{f}(t)}{\bar{g}(t)}dt, \quad (41)$$

$$S_0=-k_1g(0)[1+2e_0]^{-\frac{1}{2}}\frac{d\theta(0)}{de}, \quad (42)$$

where

$$k_1=(1+2e_0)k^*c^{-1},\quad \bar{\rho}_0=\rho_0(1+2e_0)^{-1},$$

$$\bar{f}(t)=f^*(t-e_0),\quad f^*(t)=f[c\sqrt{1+2t}-c], \quad (43)$$

$$\bar{g}(t)=g(t-e_0). \quad (44)$$

From Eqs. (8), (9), (15), (18), (38), and (40)-(44) we obtain

$$\sigma^* + \sigma^{*0} = \bar{\rho}_0 c^2 \left[1 - \frac{1}{\sqrt{1+2(e+e_0)}}\right] - \frac{m}{\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{dt}{\bar{g}(t)} -$$

$$- \frac{1}{\sqrt{1+2e_0}\,\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{\bar{f}(t)}{\bar{g}(t)}\, dt, \tag{45}$$

$$S + S_0 = -\frac{k_1 g(e)}{\sqrt{1+2(e+e_0)}} \frac{d\theta}{de} - k_2 \int_0^e \frac{f^*(t)}{g(t)} (\sqrt{1+2t} - 1)\, dt, \tag{46}$$

where $k_2 = \bar{k}/\sqrt{1 + 2e_0}$, $0 \le e_0 \le 1/2$, m is a constant of integration, $m \ge 0$, and the function g is given by Eq. (37). From Eqs. (11), (45), and (46) we have

$$\left.\frac{\partial F}{\partial (e+e_0)}\right|_{\theta=y(e,e_0)} = \bar{\rho}_0 c^2 \left[1 - \frac{1}{\sqrt{1+2(e+e_0)}}\right] - \frac{m}{\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{dt}{\bar{g}(t)} -$$

$$- \frac{1}{\sqrt{1+2e_0}\,\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{\bar{f}(t)}{\bar{g}(t)}\, dt, \tag{47}$$

$$\left.\frac{\partial F}{\partial \theta}\right|_{\theta=y(e,e_0)} = \frac{k_1 g(e)}{\sqrt{1+2(e+e_0)}} \frac{dy}{de} + k_2 \int_0^e \frac{f^*(t)}{g(t)} (\sqrt{1+2t} - 1)\, dt. \tag{48}$$

We consider the following problem: Given the function f, find a function F such that the system (47), (48) will have a solution. We seek F in the form

$$F = f_1(e+e_0) - \gamma(e+e_0)\theta - \frac{\varkappa}{2}\theta^2, \tag{49}$$

where γ and κ are constants, and f_1 is an unknown function.

Substituting Eq. (49) into Eqs. (47) and (48), we obtain the equations

$$f_1' = \gamma y + \bar{\rho}_0 c^2 \left[1 - \frac{1}{\sqrt{1+2(e+e_0)}}\right] - \frac{m}{\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{dt}{\bar{g}(t)} -$$

$$- \frac{1}{\sqrt{1+2e_0}\,\sqrt{1+2(e+e_0)}} \int_0^{e+e_0} \frac{\bar{f}(t)}{\bar{g}(t)}\, dt, \tag{50}$$

$$\gamma(e+e_0) + \varkappa y = -\frac{k_1 g(e)}{\sqrt{1+2(e+e_0)}} \frac{dy}{de} - k_2 \int_0^e \frac{f^*(t)}{g(t)} (\sqrt{1+2t} - 1)\, dt. \tag{51}$$

The following equation is deduced from Eq. (51):

$$\frac{dy}{de} = \Pi(e)\, y + \Gamma(e). \tag{52}$$

Here

$$\Pi(e) = (-\varkappa\sqrt{1+2(e+e_0)})/k_1 g(e), \tag{53}$$

$$\Gamma(e) = (-k_2\sqrt{1+2(e+e_0)} \int_0^e \frac{f^*(t)}{g(t)} (\sqrt{1+2t} - 1)\, dt -$$

$$- \gamma(e+e_0)\sqrt{1+2(e+e_0)})/k_1 g(e). \tag{54}$$

Equation (52) has the solution

$$y = \exp\left(\int_0^e \Pi(t)\, dt\right) \int_0^e \exp\left(-\int_0^t \Pi(\tau)\, d\tau\right) \Gamma(t)\, dt, \tag{55}$$

which satisfies the conditions (42) and $y(0) = 0$.

It follows from Eqs. (50) and (55) that

$$f_1(e+e_0) = \bar{\rho}_0 c^2 (e+e_0) - \bar{\rho}_0 c^2 \ln \sqrt{1+2(e+e_0)} - m \times$$

$$\times \int_0^{e+e_0} \frac{d\tau}{\sqrt{1+2\tau}} \int_0^{\tau} \frac{dt}{\bar{g}(t)} - \frac{1}{\sqrt{1+2e_0}} \int_0^{e+e_0} \frac{d\tau}{\sqrt{1+2\tau}} \int_0^{\tau} \frac{\bar{f}(t)}{\bar{g}(t)} dt -$$

$$-\gamma \int_0^{e+e_0} E(\tau)\, d\tau \left\{ \frac{\gamma}{k_1} \int_{e_0}^{\tau} \frac{t\sqrt{1+2t}}{E(t)\bar{g}(t)} dt + \frac{k_2}{k_1} \int_{e_0}^{\tau} \frac{\sqrt{1+2t}}{E(t)\bar{g}(t)} dt \int_{e_0}^{t} \frac{\bar{f}(S)}{\bar{g}(S)} (\sqrt{1+2S-2e_0} - 1)\, dS \right\}, \qquad (56)$$

where

$$E(\tau) = \exp\left(-\frac{\varkappa}{k_1} \int_{e_0}^{\tau} \frac{\sqrt{1+2t}}{\bar{g}(t)} dt \right). \qquad (57)$$

Thus, if the function f is given, the function F is determined from Eq. (49), in which the function f_1 is calculated according to Eq. (56). The stress σ^* and the entropy S are determined from Eqs. (11), (49), and (56), in which the function θ is expressed by Eq. (55). The dependence of the wave velocity on the initial state and on the dry friction law is given by Eq. (41).

NOTATION

T, absolute temperature of the medium at a given time t; T_0, value of T at initial time t_0; S, entropy of the medium; x, Lagrangian coordinates of a point of the medium; ρ_0, material density of the medium in the natural state; σ^*, x-component of the generalized stress tensor; u, displacement of a point of the medium along the x-axis; v, velocity of a point of the medium in the x-direction; e, strain of the medium along the x-axis.

LITERATURE CITED

1. Yu. K. Éngel'brekht and U. K. Nigul, Nonlinear Strain Waves [in Russian], Moscow (1981).
2. A. N. Guz', Elastic Waves in Initially Stressed Bodies [in Russian], Vol. 1, Kiev (1986).
3. V. V. Novozhilov, Theory of Elasticity [in Russian], Leningrad (1958).
4. A. E. Green and J. E. Adkins, Large Elastic Deformations, 2nd ed., Oxford Univ. Press, London (1971).
5. M. D. Martynenko and Nguen Dang Bik, Dokl. Akad. Nauk B. SSR, 34, No. 9, 802-807 (1990).
6. M. D. Martynenko and Nguen Dang Bik, Vestsi Akad. Nauk B. SSR, Ser. Fiz. Tekh. Nauk, No. 4, 42-47 (1990).